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Math 12 Enriched: Section 3.2 Factoring Polynomial Functions

1. Find the Quotient, remainder and write the division statement:

a) $y = x^2 - 5x + 2 \div (x-3)$

Remainder $= (3)^2 - 5(3) + 2 = -4$

$$\begin{array}{r|rrr} 3 & 1 & -5 & 2 \\ & & 3 & -6 \\ \hline & 1 & -2 & -4 \end{array}$$

remainder

OR

$$\begin{array}{r|l} & x-2 \\ x-3 & x^2-5x+2 \\ & -(x^2-3x) \\ & -2x+2 \\ & -(-2x+6) \\ & -4 \end{array}$$

$\boxed{x^2 - 5x + 2 = (x-3)(x-2) - 4}$

Division statement: dividend = (divisor)(quotient) − remainder

b) $y = x^3 + x^2 - 10x + 8 \div (x-2)$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -10 & 8 \\ & & 2 & 6 & -8 \\ \hline & 1 & 3 & -4 & 0 \end{array}$$

Rem = 0

$\boxed{x^3 + x^2 - 10x + 8 = (x-2)(x^2+3x-4)}$

divisor quotient

c) $y = x^3 + x^2 - 2x + 3 \div (x+1)$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -2 & 3 \\ & & -1 & 0 & 2 \\ \hline & 1 & 0 & -2 & 5 \end{array}$$

Remander = 5

$\boxed{x^3 + x^2 - 2x + 3 = (x+1)(x^2-2) + 5}$

quotient $= x^2 - 2$

d) $y = 3x^3 - 5x^2 + 2x + 8 \div (2x-1)$

$$\begin{array}{r|rrrr} \frac{1}{2} & 3 & -5 & 2 & 8 \\ & & \frac{3}{2} & -\frac{7}{4} & \frac{1}{8} \\ \hline & 3 & -\frac{7}{2} & \frac{1}{4} & \frac{65}{8} \end{array}$$

DIVIDE BY COEFFICIENT OF x in divisor.

$\boxed{3x^3 - 5x^2 + 2x + 8 = (2x-1)\left(\frac{3}{2}x^2 - \frac{7}{4}x + \frac{1}{8}\right) + \frac{65}{8}}$

e) $y = 2x^3 + x^2 + 4x - 7 \div (x-4)$

$$\begin{array}{r|rrrr} 4 & 2 & 1 & 4 & -7 \\ & & 8 & 36 & 160 \\ \hline & 2 & 9 & 40 & 153 \end{array}$$

$\boxed{2x^3 + x^2 + 4x - 7 = (x-4)(2x^2+9x+40) + 153}$

f) $y = x^4 - 3x^3 + 2x^2 - 5x - 1 \div (2x+3)$

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 1 & -3 & 2 & -5 & -1 \\ & & -\frac{3}{2} & \frac{27}{4} & -\frac{105}{8} & \frac{435}{16} \\ \hline & 1 & -\frac{9}{2} & \frac{35}{4} & -\frac{145}{8} & \frac{419}{16} \end{array}$$

$\boxed{x^4 - 3x^3 + 2x^2 - 5x - 1 = (2x+3)\left(\frac{x^3}{2} - \frac{9}{4}x^2 + \frac{35}{8}x - \frac{145}{16}\right) + \frac{419}{16}}$

2. Determine each value of k:

a) When $x^3 + kx^2 + 2x - 3$ is divided by x+2, the remainder is 1.

$$\begin{array}{r|rrrr} -2 & 1 & k & 2 & -3 \\ & & -2 & -2k+4 & 4k-12 \\ \hline & 1 & k-2 & -2k+6 & 4k-15 \end{array}$$

$k=4$

OR

$f(-2) = 1$

$(-2)^3 + k(2)^2 + 2(-2) - 3 = 1$

$4k = 16$

$\boxed{k=4}$

b) When $x^4 - kx^3 - 2x^2 + x + 4$ is divided by x-3, the remainder is 16.

$f(3) = 16$

$(3)^4 - k(3)^3 - 2(3)^2 + (3) + 4 = 16$

$-27k = -54$

$\boxed{k=2}$

c) When $2x^3 - 3x^2 + kx - 1$ is divided by x-1, the remainder is 1.

$f(1) = 1$

$2 - 3 + k - 1 = 1$

$\boxed{k=3}$

d) When $2x^4 + kx^2 - 3x + 5$ is divided by x-2, the remainder is 3.

$f(2) = 3$

$2(16) + 4k - 6 + 5 = 3$

$4k = -28 \Rightarrow \boxed{k=-7}$

e) When $x^3 + kx^2 - 2x - 7$ is divided by x+1, the remainder is 5.

$f(-1) = 5$

$-1 + k + 2 - 7 = 5$

$\boxed{k=11}$

f) When $kx^3 + 2x^2 - x + 3$ is divided by x+1, the remainder is 4.

$f(-1) = 4$

$-k + 2 + 1 + 3 = 4$

$-k = -2$

$\boxed{k=2}$

3. How do you know if a binomial is a factor of a polynomial? Explain it using your own words:

If a binomial is a factor of a polynomial, then they must share an x-intercept. Therefore, plugging in the binomial's root inside the polynomial must yield a value of 0 if they are factors of each other.

4. Given $f(x)=2x^3-5x^2-x+6$, which of the following binomials is a factor $f(x)$?

i) $\cancel{(x+2)}$ ii) $\cancel{(x-1)}$ iii) $\cancel{(2x+3)}$ iv) $(x+1)$ v) $(2x-3)$ vi) $\cancel{(x+3)}$ vii) $(x-2)$ viii) $\cancel{(2x-1)}$

$x=-2$

$2(-2)^3-5(-2)^2-(-2)+6\neq0$

$2(-\frac{3}{2})^3-5(-\frac{3}{2})^2-(-\frac{3}{2})+6=-10.5$

$2(1.5)^3-5(1.5)^2-(1.5)+6=0$ ✓

$2(2)^3-5(2)^2-(2)+6=0$

5. How do you determine which factors to divide your function by when converting it to factor form? Explain using your own words. Provide an example:

The factors are likely factors of the constant in the polynomial, so starting with those is usually best.

Example: $2x^3-3x^2-8x+12$

I would start by dividing by factors of $\frac{12}{2}=6$

±1 ±2 ±3 ±6

works!!!

6. Use the factor theorem to convert each function to factored form: YOU MAY HAVE DOUBLE ROOTS!

a) $f(x)=2x^3-3x^2-8x+12$

$f(2)=16-12-16+12=0$

FACTOR: $(x-2)$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -8 & 12 \\ & & 4 & 2 & -12 \\ \hline & 2 & 1 & -6 & 0 \end{array}$$

$2x^2+x-6$

$2 \quad -3$

$f(x)=(x-2)(2x^2+x-6)$

$\boxed{f(x)=(x-2)(2x-3)(x+2)}$

b) $f(x)=2x^4-15x^3+36x^2-35x+12$

$f(1)=0$

factor: $(x-1)$

$$\begin{array}{r|rrrrr} 1 & 2 & -15 & 36 & -35 & 12 \\ & & 2 & -13 & 23 & -12 \\ \hline & 2 & -13 & 23 & -12 & 0 \end{array}$$

$f(x)=(x-1)(2x^3-13x^2+23x-12)$

$f(\frac{3}{2})=0$ factor: $(x-\frac{3}{2})$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -13 & 23 & -12 \\ & & 3 & -15 & 12 \\ \hline & 2 & -10 & 8 & 0 \end{array}$$

$f(x)=(x-1)(x-\frac{3}{2})(2x^2-10x+8)$

$f(x)=(x-1)(x-\frac{3}{2})(2)(x-1)(x-4)$

$\boxed{f(x)=2(x-1)^2(x-4)(x-\frac{3}{2})}$

c) $f(x)=20x^3+17x^2-40x+12$

$f(-2)=0$

$$\begin{array}{r|rrrr} -2 & 20 & 17 & -40 & 12 \\ & & -40 & 46 & -12 \\ \hline & 20 & -23 & 6 & 0 \end{array}$$

$f(x)=(x+2)(20x^2-23x+6)$

$4 \quad -2$

$5 \quad -3$

$\boxed{f(x)=(x+2)(4x-3)(5x-2)}$

d) $f(x)=x^3+9x^2+26x+24$

$f(-2)=0$

$$\begin{array}{r|rrrr} -2 & 1 & 9 & 26 & 24 \\ & & -2 & -14 & -24 \\ \hline & 1 & 7 & 12 & 0 \end{array}$$

$f(x)=(x+2)(x^2+7x+12)$

$=\boxed{(x+2)(x+3)(x+4)}$

e) $f(x)=2x^3+x^2-25x+12$

$f(3)=0$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & 0 \end{array}$$

$f(x)=(x-3)(2x^2+7x-4)$

$2 \quad -\frac{1}{4}$

$\boxed{f(x)=(x-3)(2x-1)(x+4)}$

f) $f(x)=2x^4-7x^3+9x^2-5x+1$

$f(1)=0$

$$\begin{array}{r|rrrrr} 1 & 2 & -7 & 9 & -5 & 1 \\ & & 2 & -5 & 4 & -1 \\ \hline & 2 & -5 & 4 & -1 & 0 \end{array}$$

$f(x)=(x-1)\underbrace{(2x^3-5x^2+4x-1)}_{\text{root: } x-1}$

$$\begin{array}{r|rrrr} 1 & 2 & -5 & 4 & -1 \\ & & 2 & -3 & 1 \\ \hline & 2 & -3 & 1 & 0 \end{array}$$

$f(x)=(x-1)^2(2x^2-3x+1) \Rightarrow \boxed{f(x)=(x-1)^2(x-1)(2x-1)}$

$2 \quad -1$

7. When kx^3+mx^2+x-2 is divided by $x-1$, the remainder is 6. When this polynomial is divided by $x+2$, the remainder is 12. Solve for "k" and "m".

$f(1)=k+m+1-2=6$

$\Rightarrow k+m=7$

$f(-2)=-8k+4m-2-2=12$

$\Rightarrow -8k+4m=16$

$\Rightarrow 2k-m=-4$

add

$k+m+2k-m=7-4 \Rightarrow 3k=3 \Rightarrow k=1 \quad m=7-k=6$

$\boxed{(k,m)=(1,6)}$

8. Factor completely: $8r^2+6rs-12rs-9s^2$

Approach 1

$2r(4r+3s)-3s(4r+3s) \Rightarrow \boxed{(4r+3s)(2r-3s)}$

Approach 2

$8r^2-6rs-9s^2$

2 +3 6

4 −3 −12

$\Rightarrow \boxed{(4r+3s)(2r-3s)}$

9. If r_1, r_2, r_3, r_4 are the roots of $x^4-9x^2+2=0$, what is the value of $(1+r_1)(1+r_2)(1+r_3)(1+r_4)$?

$(-x+r_1)(-x+r_2)(-x+r_3)(-x+r_4)=x^4-9x^2+2$

if $x=-1$:

$(1+r_1)(1+r_2)(1+r_3)(1+r_4)=(-1)^4-9(-1)^2+2=\boxed{-6}$

10. If $f(2x)=x^2+4x+1$, what are al values of "t" for which $f\left(\frac{t}{2}\right)=\frac{-11}{4}$, where "f" represents a function?

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$f(2x)=x^2+4x+1$

$x=\frac{t}{4}$

$f\left(\frac{t}{2}\right)=\left(\frac{t}{4}\right)^2+4\left(\frac{t}{4}\right)+1=\frac{-11}{4}$

$\frac{t^2}{16}+t=\frac{-15}{4} \Rightarrow t^2+16t+60=0 \Rightarrow (t+6)(t+10)=0 \Rightarrow \boxed{\begin{array}{l} t_1=-6 \\ t_2=-10 \end{array}}$

6, 10

11. There are two real values of "r" for which $x^4-x^3-18x^2+52x+k$ has a factor of the form $x-r$. One of these values is $r=2$. What is the other value of "r"?

$f(2)$

$2^4-2^3-18(2)^2+52(2)+k=0$

$k=-40$

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -18 & 52 & -40 \\ & \downarrow & 2 & 2 & -32 & 40 \\ \hline & 1 & 1 & -16 & 20 & 0 \end{array}$$

$f(x)=(x-2)(x^3+x^2-16x+20)$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -16 & 20 \\ & \downarrow & 2 & 6 & -20 \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

$f(x)=(x-2)^2(x^2+3x-10) \Rightarrow f(x)=(x-2)^3(x+5)$

$\boxed{r=-5}$

12. For what rational number "c" do the equations $x^3+cx^2+3=0$ and $x^2+cx+1=0$ have a common solution?

$f(x)=x^3+cx^2+3=(x-r_1)(x-r_2)(x-a)$

$g(x)\ x^2+cx+1=(x-r_3)(x-a)$

$f(a)=0 \Rightarrow a^3+ca^2+3=0$

$g(a)=0 \Rightarrow a^2+ca+1=0 \Rightarrow a^3+a^2c+a=0$

subract: $a-3=0 \Rightarrow a=3$

$(3)^2+3c+1=0 \Rightarrow \boxed{c=-\frac{10}{3}}$

13. Determine the exact value of the only real root of the equation: $x^3+6x^2+12x+24=0$. (No calculators!!)

$x^3+6x^2+12x+8+16=0$

1 3 3 1

$(x+2)^3+16=0$

$x+2=\sqrt[3]{16}=-2\sqrt[3]{2}$

$\boxed{x=-2\sqrt[3]{2}-2}$

Pascal's Triangle

Only one root!!!

YOU MISSED A ROOT!!!

14. Determine all the solutions to the systems of equations: $x^2+y^2+x+y=12$ and $xy+x+y=3$

$xy+x+y=3$

$x+y=3-xy$

$x^2+2xy+y^2=9-6xy+x^2y^2$

$x^2+y^2=x^2y^2-8xy+9$

$x^2+y^2+x+y-(xy+x+y)=12-3$

$x^2+y^2-xy=9$

$x^2+y^2=9+xy$

$x^2y^2-8xy+9-9+xy \Rightarrow x^2y^2-9xy=0 \Rightarrow xy=0$ or, $xy=9$ (now plug back in)

FINAL

$0+x+0=3$

$(x_1,y_1)=\boxed{(3,0)},\boxed{(0,3)}$

$xy+x+y=3$

$9+x+\frac{9}{x}=3 \Rightarrow x^2+6x+9=0 \Rightarrow x=-3, y=-3$

$\boxed{x_2, y_2=-3,-3}$

15. One real root of the equation $x^8+x^6+x^4+x^2=340$ is $x=2$. What is the only other real root of this equation?

If $x=2$ is a root, the $x=-2$ must also be a root since all of our exponents are even and therefore eliminate the negative sign.

$\boxed{r_2=-2}$

16. Let "r" be a root of $x^4-x^3+x^2-x+1=0$. What is the value of $r^{40}-r^{30}+r^{20}-r^{10}+1$?

$S_n=\dfrac{a(1-r^n)}{1-r}$ $n=5$, $a=1$, $r=-x$

$r^{40}-r^{30}+r^{20}-r^{10}+1=1-1+1-1+1=\boxed{1}$

$f(x)=S_n=\dfrac{(1+x^5)}{1+x} \Rightarrow \dfrac{1+r^5}{1+r}=0$

$\Rightarrow r^5+1=0$

$r=-1$ extraneous.. but usable

17. Solve the following inequalities:

i) $x^4-10x^3+35x^2-50x+24>0$

$f(1)=0$

1	1	−10	35	−50	24
		1	−9	26	−24
	1	−9	26	−24	0

$y=(x-1)(x^3-9x^2+26x-24)$

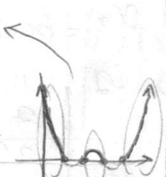
$f(2)=0$

2	1	−9	26	−24
		2	−14	24
	1	−7	12	0

$y=(x-1)(x-2)(x^2-7x+12)$

$\boxed{y=(x-1)(x-2)(x-3)(x-4)}$

$\boxed{x<1 \cup 2<x<3 \cup 4<x}$



ii) $x^4+6x^3-13x^2-66x+72\le 0$

$f(1)=0$

1	1	6	−13	−66	72
		1	7	−6	−72
	1	7	−6	−72	0

$y=(x-1)(x^3+7x^2-6x-72)$

$f(3)=0$

3	1	7	−6	−72
		3	30	72
	1	10	24	0

$y=(x-1)(x^2+10x+24)(x-3)$

$y=(x-1)(x-2)(x+12)(x-3)$

$\boxed{x\le -12 \cup -1<x<2 \cup x\ge 3}$

18. Challenge: Let $P(x)=(x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x))=P(x)\bullet R(x)$?

② $R(x)$ needs to be quadratic

① $P(Q(x))=(Q(x)-1)(Q(x)-2)(Q(x)-3)=(x-1)(x-2)(x-3)\cdot R(x)$

$P(Q(1))=(Q(1)-1)(Q(1)-2)(Q(1)-3)=P(1)\cdot R(1)=0$

$P(Q(2))=(Q(2)-1)(Q(2)-2)(Q(2)-3)=P(2)\cdot R(2)=0$

$P(Q(3))=(Q(3)-1)(Q(3)-2)(Q(3)-3)=P(3)\cdot R(3)=0$

for these to be zero, $Q(1)$, $Q(2)$, or $Q(3)$ must be 1, 2, or 3 each.

③ So $3\times3\times3=27$ "$Q(x)$"s are possible.

④ however, we have also counted linear "$Q(x)$"s, so we must get rid of them:

5 bad cases:

$Q(1)=Q(2)=Q(3)=1 \Rightarrow Q(x)=1$

" " " $=2 \Rightarrow Q(x)=2$

" " " $=3 \Rightarrow Q(x)=3$

$Q(1), Q(2), Q(3)=1,2,3$ respectively $\Rightarrow Q(x)=x$

$Q(1), Q(2), Q(3)=3,2,1$ respectively $\Rightarrow Q(x)=4-x$

⑤ $27-5=\boxed{22}$

Find the remainder when $x^{81}+x^{49}+x^{25}+x^{9}+x$ is divided by x^3-x.

$$\frac{x^{81}+x^{49}+x^{25}+x^{9}+x}{x(x+1)(x-1)} = \frac{x^{80}+x^{48}+x^{24}+x^{8}+1}{(x+1)(x-1)}$$

1 | 1 0 0 0 ... 0 1 ... 1 ... 1 ... 1
1 1 1 1 1 2 2 2 2 3 3 3 4 4 4 5

$\Longrightarrow$

32 24 16 8 1

-1 | 1 1 1 1 1 ... 2 2 2 ... 3 3 3 ... 4 4 4 ... 5
-1 0 -1 0 -2 0 0 -3 0 0 -4 0 0
1 0 1 0 1 0 1 0 2 0 2 0 3 0 3 0 4 0 4 0 $\boxed{5}$

Remainder = 5

The polynomial $p(x)$ satisfies $p(-x) = -p(x)$. When $p(x)$ is divided by $x-3$ the remainder is 6. Find the remainder when $p(x)$ is divided by x^2-9.

$P(3)=6$

$P(-x)=-P(x) \Rightarrow P(-3)=-6$

$P(x)=(x-3)(x+3)+ax+b$

$P(3)=3a+b=6$

$P(-3)=-3a+b=-6$

$6a=12$

$a=2$

$b=0$

$P(x)=(x-3)(x+3)+2x$

$P(x)=x^2+2x-9$

$$\begin{array}{r} 1 \\ x^2-9 \,\big)\, x^2+2x-9 \\ -(x^2-9) \\ \hline 2x \end{array}$$

$\boxed{\text{Remainder} = 2x}$